Summer Bootcamp: Linear Algebra

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Contents

1 Motivation and Applications

- Linear programming (LP) is a key tool for optimizing outcomes under linear constraints.
- Applications include production scheduling, transportation, finance, diet planning, resource allocation, and more.
- This lecture builds the mathematical foundation necessary to understand LP.

2 Scalars, Vectors, and Matrices

2.1 Scalars

A scalar is a single numerical value (e.g., $a \in \mathbb{R}$).

2.2 Vectors

A vector is an ordered list of numbers:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Vector Operations:

- **Addition:** Given two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$, their sum is:

$$\mathbf{a} + \mathbf{b} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + b_n \end{bmatrix}$$

Example: Let $\mathbf{a} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, then:

$$\mathbf{a} + \mathbf{b} = \begin{bmatrix} 1+4 \\ 2+5 \\ 3+6 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix}$$

- **Scalar multiplication:** For a scalar $\lambda \in \mathbb{R}$ and vector $\mathbf{a} \in \mathbb{R}^n$:

$$\lambda \mathbf{a} = \begin{bmatrix} \lambda a_1 \\ \lambda a_2 \\ \vdots \\ \lambda a_n \end{bmatrix}$$

Example: Let $\lambda = 3$, $\mathbf{a} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$, then:

$$3\mathbf{a} = \begin{bmatrix} 6 \\ -3 \\ 12 \end{bmatrix}$$

- **Dot product:** For vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$, the dot product is:

$$\mathbf{a}^\top \mathbf{b} = \sum_{i=1}^{n} a_i b_i$$

Example: Let $\mathbf{a} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 4 \\ -5 \\ 6 \end{bmatrix}$, then:

$$\mathbf{a}^\top \mathbf{b} = (1)(4) + (2)(-5) + (3)(6) = 4 - 10 + 18 = 12$$

- **Norm (Euclidean length):** The norm of a vector $\mathbf{a} \in \mathbb{R}^n$ is:

$$\|\mathbf{a}\| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$$

Example: For $\mathbf{a} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$:

$$\|\mathbf{a}\| = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$$

2.3 Matrices

A matrix is a rectangular array:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Matrix Operations:

- **Matrix addition and scalar multiplication:** Given two matrices $A, B \in \mathbb{R}^{m \times n}$, and a scalar $\lambda \in \mathbb{R}$, we define:

$$A + B = [a_{ij} + b_{ij}], \quad \lambda A = [\lambda a_{ij}]$$

Example: Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}, \quad \lambda = 2$

Then:

$$A + B = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}, \quad \lambda A = 2 \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$$

- **Matrix-vector multiplication:** Given $A \in \mathbb{R}^{m \times n}$, $\mathbf{x} \in \mathbb{R}^n$:

$$A\mathbf{x} = \begin{bmatrix} \sum_{j=1}^{n} a_{1j} x_j \\ \sum_{j=1}^{n} a_{2j} x_j \\ \vdots \\ \sum_{j=1}^{n} a_{mj} x_j \end{bmatrix}$$

Example: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 5 \\ 6 \end{bmatrix} \Rightarrow A\mathbf{x} = \begin{bmatrix} 1(5) + 2(6) \\ 3(5) + 4(6) \end{bmatrix} = \begin{bmatrix} 17 \\ 39 \end{bmatrix}$

- **Matrix-matrix multiplication:** For $A \in \mathbb{R}^{m\times n}, B \in \mathbb{R}^{n\times p}$, their product $C = AB \in \mathbb{R}^{m\times p}$ is:

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

  Example:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$$

$$AB = \begin{bmatrix} (1)(2) + (2)(1) & (1)(0) + (2)(3) \\ (3)(2) + (4)(1) & (3)(0) + (4)(3) \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 10 & 12 \end{bmatrix}$$

- **Transpose:** The transpose of a matrix $A \in \mathbb{R}^{m\times n}$ is $A^\top \in \mathbb{R}^{n\times m}$, where:

$$(A^\top)_{ij} = A_{ji}$$

Determinant of a Matrix

The determinant is a scalar associated with a square matrix. It tells us:

- Whether the matrix is invertible
- The signed volume scaling factor of the associated linear transformation

2×2 Case

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

Example:

$$\det \begin{bmatrix} 3 & 2 \\ 5 & 4 \end{bmatrix} = (3)(4) - (2)(5) = 12 - 10 = 2$$

3×3 Case

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = aei + bfg + cdh - ceg - bdi - afh$$

Example:

$$\det \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 0 & 6 \end{bmatrix} = 1(4)(6) + 2(5)(1) + 3(0)(0) - 3(4)(1) - 2(0)(6) - 1(5)(0) = 24 + 10 - 12 = 22$$

General $n \times n$: Cofactor Expansion

Let $A = [a_{ij}] \in \mathbb{R}^{n\times n}$. The determinant can be expanded along any row i:

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \cdot \det(M_{ij})$$

where M_{ij} is the minor matrix obtained by removing row i and column j.

Example:

$$\det \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = 1 \cdot \det \begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix} - 2 \cdot \det \begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix} + 3 \cdot \det \begin{bmatrix} 4 & 5 \\ 7 & 8 \end{bmatrix}$$

Determinant Properties

- $\det(I) = 1$
- $\det(\lambda A) = \lambda^n \det(A)$
- $\det(AB) = \det(A)\det(B)$
- $\det(A^\top) = \det(A)$
- $\det(A) = 0$ if and only if A is not invertible

3 Linear Systems and Gaussian Elimination

$$A\mathbf{x} = \mathbf{b}$$

Geometric Interpretation: Hyperplanes in $\mathbb{R}^n$

Each linear equation in the system $A\mathbf{x} = \mathbf{b}$ represents a **hyperplane** in $\mathbb{R}^n$. A hyperplane is an $(n-1)$-dimensional flat subspace that divides the space into two halves.

Definition: A hyperplane is defined as:

$$\mathbf{a}^\top \mathbf{x} = b \quad \text{or} \quad a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where:

- $\mathbf{a}$ is the normal vector to the hyperplane,
- $\mathbf{x} \in \mathbb{R}^n$ is the variable vector,
- $b \in \mathbb{R}$ is a scalar constant.

Solution of a linear system geometrically corresponds to the *intersection* of all such hyperplanes:

- A unique solution exists if the hyperplanes intersect at a single point.
- Infinitely many solutions arise if they intersect along a line, plane, or higher-dimensional subspace.
- No solution exists if the hyperplanes do not all intersect—i.e., the system is inconsistent.

Example:
Consider:

$$\begin{cases} x_1 + x_2 = 4 \\ 2x_1 - x_2 = 1 \end{cases}$$

Each equation is a line (a 1D hyperplane in $\mathbb{R}^2$). Their intersection is the point where both lines meet.

Distance Between Two Hyperplanes

Consider two parallel hyperplanes in $\mathbb{R}^n$:

$$H_1 : \mathbf{a}^\top \mathbf{x} = b_1, \quad H_2 : \mathbf{a}^\top \mathbf{x} = b_2$$

Both hyperplanes share the same normal vector $\mathbf{a} \in \mathbb{R}^n$ and differ only in the constant term.

Formula for the distance between the two hyperplanes:

$$\text{Distance} = \frac{|b_2 - b_1|}{\|\mathbf{a}\|}$$

Explanation:

- $\|\mathbf{a}\|$ is the Euclidean norm (length) of the normal vector.
- The formula computes the perpendicular distance between the hyperplanes.
- This is meaningful only when the hyperplanes are parallel.

Example: Let:

$$H_1 : 2x_1 + x_2 = 4, \quad H_2 : 2x_1 + x_2 = 1$$

Then:

$$\mathbf{a} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \|\mathbf{a}\| = \sqrt{2^2 + 1^2} = \sqrt{5}$$

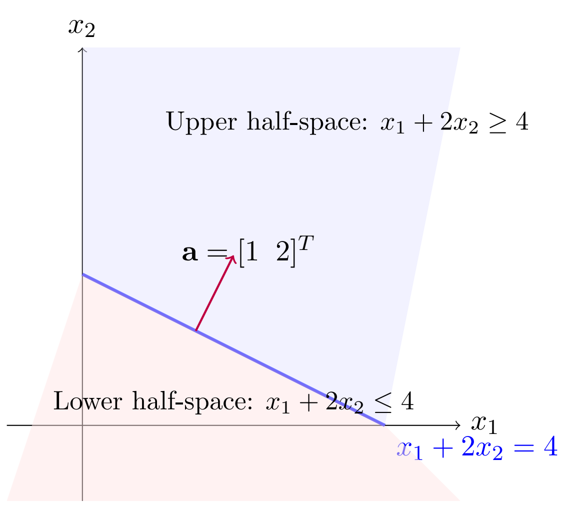
$$\text{Distance} = \frac{|4 - 1|}{\sqrt{5}} = \frac{3}{\sqrt{5}} \approx 1.34$$

Half-Spaces

Each inequality (such as $\mathbf{a}^\top \mathbf{x} \leq b$) defines a **half-space**, which includes all points on one side of the hyperplane:

- **Lower half-space:** $\mathbf{a}^\top \mathbf{x} \leq b$
- **Upper half-space:** $\mathbf{a}^\top \mathbf{x} \geq b$

In LP: Feasible regions are formed by the intersection of multiple half-spaces. Solutions lie on or within the boundaries (i.e., hyperplanes).



Row Operations (Gaussian Elimination)

- Swap two rows
- Multiply a row by a nonzero constant
- Add a multiple of one row to another row

These row operations transform the system into upper triangular or row-echelon form, allowing for systematic solution (back-substitution).

Example of Gaussian Elimination

Solve:

$$\begin{cases} 2x_1 + x_2 - x_3 = 8 \\ -3x_1 - x_2 + 2x_3 = -11 \\ -2x_1 + x_2 + 2x_3 = -3 \end{cases}$$

4 Linear Independence and Rank

Linear Dependence and Independence

A set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is:

- **Linearly independent** if the only solution to

  $$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k = \mathbf{0}$$

  is $a_1 = a_2 = \cdots = a_k = 0$.
- **Linearly dependent** if there exist scalars (not all zero) such that the above combination equals $\mathbf{0}$.

Example: Let

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$$

Then:

$$\mathbf{v}_2 = 2\mathbf{v}_1 \Rightarrow \text{Linearly dependent}$$

Now consider:

$$\mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

The set $\{\mathbf{v}_1, \mathbf{v}_3\}$ is linearly independent.

Geometry of a Linear System

Each linear equation defines a geometric object:

- In $\mathbb{R}^2$: a line
- In $\mathbb{R}^3$: a plane
- In $\mathbb{R}^n$: a hyperplane

The solution of a linear system corresponds to the intersection of these objects:

- Two intersecting lines $\Rightarrow$ unique solution
- Parallel lines $\Rightarrow$ no solution
- Coincident lines $\Rightarrow$ infinitely many solutions

Matrix Rank

The **rank** of a matrix is the number of linearly independent rows or columns.

Interpretation:

- Rank tells us the number of dimensions spanned by the rows or columns.
- A matrix with full rank (i.e., rank = number of columns) has linearly independent columns and defines an invertible linear transformation.
- A lower rank implies redundancy or dependence among the rows or columns.

Example 1: Full Rank

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \text{Rank}(A) = 2$$

Example 2: Rank Deficient

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix} \quad \text{Rank}(B) = 1 \quad \text{(rows are multiples of each other)}$$

How to Determine Rank (Numerically)

- Use Gaussian elimination to reduce the matrix to row echelon form
- Count the number of non-zero rows

Interpretation of Rank in Solving Linear Systems

Let:

$$A\mathbf{x} = \mathbf{b}, \quad A \in \mathbb{R}^{m \times n}, \quad \text{and augmented matrix } [A \mid \mathbf{b}]$$

Define:

- $\text{rank}(A)$: number of linearly independent rows of coefficient matrix
- $\text{rank}([A \mid \mathbf{b}])$: rank of augmented matrix

- n: number of variables (columns of A)

Classification of Solutions:

- **Unique solution:**
  - $\text{rank}(A) = \text{rank}([A \mid \mathbf{b}]) = n$
  - All variables are basic (no free variables)
  - Geometric: intersection of hyperplanes is a single point
- **Infinitely many solutions:**
  - $\text{rank}(A) = \text{rank}([A \mid \mathbf{b}]) < n$
  - Some variables are free (under-determined system)
  - Geometric: hyperplanes intersect along a line, plane, etc.
- **No solution:**
  - $\text{rank}(A) < \text{rank}([A \mid \mathbf{b}])$
  - Inconsistent system (e.g., leads to contradiction like $0 = 1$)
  - Geometric: hyperplanes do not share a common intersection (e.g., parallel)

5 Matrix Inverse

A square matrix $A \in \mathbb{R}^{n \times n}$ is invertible (non-singular) if there exists a matrix A^{-1} such that:

$$AA^{-1} = A^{-1}A = I$$

where I is the identity matrix of size $n \times n$.

2×2 Inverse (Closed Formula)

For a 2×2 matrix:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \quad \text{if } ad - bc \neq 0$$

Example: Let

$$A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}, \quad \det(A) = 2 \cdot 3 - 1 \cdot 5 = 6 - 5 = 1$$

$$A^{-1} = \frac{1}{1} \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$$

Inverse via Gaussian Elimination

We augment the matrix with the identity and row-reduce:

$$[A \mid I] = \left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 5 & 3 & 0 & 1 \end{array}\right]$$

Step 1: Make the pivot in row 1 equal to 1

$$R_1 \leftarrow \frac{1}{2}R_1 \Rightarrow \left[\begin{array}{cc|cc} 1 & 0.5 & 0.5 & 0 \\ 5 & 3 & 0 & 1 \end{array}\right]$$

Step 2: Eliminate below

$$R_2 \leftarrow R_2 - 5R_1 \Rightarrow \left[\begin{array}{cc|cc} 1 & 0.5 & 0.5 & 0 \\ 0 & 0.5 & -2.5 & 1 \end{array}\right]$$

Step 3: Make pivot in row 2 equal to 1

$$R_2 \leftarrow 2R_2 \Rightarrow \left[\begin{array}{cc|cc} 1 & 0.5 & 0.5 & 0 \\ 0 & 1 & -5 & 2 \end{array}\right]$$

Step 4: Eliminate above

$$R_1 \leftarrow R_1 - 0.5R_2 \Rightarrow \left[\begin{array}{cc|cc} 1 & 0 & 3 & -1 \\ 0 & 1 & -5 & 2 \end{array}\right]$$

Thus,

$$A^{-1} = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$$

Conclusion

Both methods yield the same result. The inverse exists because $\det(A) \neq 0$. You can use the formula for 2×2 matrices, but for larger matrices, Gaussian elimination is more general and scalable.

6 Standard Form of Linear Programming

A linear programming (LP) problem is in **standard form** when it is written as:

$$\begin{aligned} \text{Minimize} \quad & \mathbf{c}^\top \mathbf{x} \\ \text{Subject to} \quad & A\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq 0 \end{aligned}$$

where:

- $\mathbf{x} \in \mathbb{R}^n$: vector of decision variables
- $A \in \mathbb{R}^{m \times n}$: constraint coefficient matrix
- $\mathbf{b} \in \mathbb{R}^m$: right-hand side constants
- $\mathbf{c} \in \mathbb{R}^n$: objective function coefficients

Example (already in standard form)

$$\begin{aligned} \text{Minimize} \quad & 3x_1 + 5x_2 \\ \text{Subject to} \quad & x_1 + x_2 = 7 \\ & 2x_1 + 4x_2 = 12 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Converting to Standard Form

Many problems do not start in standard form. We use the following strategies to convert them:

1. **Inequality to Equality:** Convert $\leq$ constraints into equalities using **slack variables**. Example:

   $$x_1 + 2x_2 \leq 8 \quad \Rightarrow \quad x_1 + 2x_2 + s_1 = 8, \quad s_1 \geq 0$$

   where s_1 is the slack variable.

2. **Maximization to Minimization:** Multiply the objective by -1. Example:

   $$\text{Maximize } 4x_1 + 6x_2 \quad \Rightarrow \quad \text{Minimize } -4x_1 - 6x_2$$

3. **Unrestricted Variables:** A variable x_j unrestricted in sign can be written as the difference of two nonnegative variables:

   $$x_j = x_j^+ - x_j^-, \quad \text{with } x_j^+, x_j^- \geq 0$$

Example: Converting to Standard Form

$$\begin{aligned} \text{Maximize} \quad & z = 3x_1 + 5x_2 \\ \text{Subject to} \quad & x_1 + 2x_2 \leq 8 \\ & 4x_1 + x_2 \geq 4 \\ & x_1 \text{ unrestricted}, \quad x_2 \geq 0 \end{aligned}$$

Step 1: Convert to minimization:

$$\text{Minimize } -3x_1 - 5x_2$$

Step 2: Add slack/surplus variables:

$$\begin{aligned} x_1 + 2x_2 + s_1 = 8 \quad (s_1 \geq 0) \\ 4x_1 + x_2 - s_2 = 4 \quad (s_2 \geq 0) \end{aligned}$$

Step 3: Handle unrestricted variable:

$$x_1 = x_1^+ - x_1^-, \quad x_1^+, x_1^- \geq 0$$

Now the problem is fully in standard form with decision variables:

$$x_1^+, x_1^-, x_2, s_1, s_2 \geq 0$$

7 Feasible Regions and Inequalities

A system of linear inequalities can be written as:

$$A\mathbf{x} \leq \mathbf{b}$$

.

Feasibility

- The solution set of the inequality system is called the **feasible region**.
- It includes all vectors $\mathbf{x} \in \mathbb{R}^n$ that satisfy all constraints.
- The feasible region may be:
  - **Bounded:** contained within a finite area (e.g., a polygon).
  - **Unbounded:** extends infinitely in some direction.
  - **Empty:** no solution exists if the constraints are inconsistent.

Example:

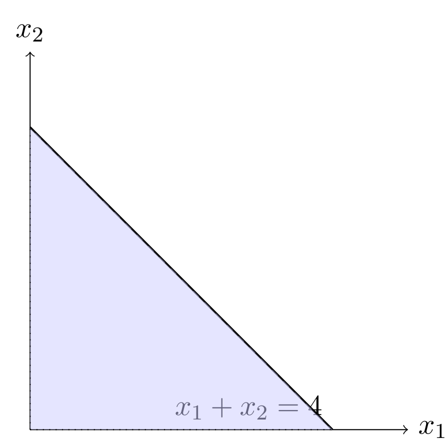
Consider the following system of inequalities:

$$\begin{aligned} x_1 + x_2 &\leq 4 \\ x_1 &\geq 0 \\ x_2 &\geq 0 \end{aligned}$$

These represent:

- The line $x_1 + x_2 = 4$ and the half-plane below it.
- The first quadrant (because $x_1, x_2 \geq 0$).

The feasible region is the triangle bounded by the axes and the line $x_1 + x_2 = 4$.



Geometric Interpretation

Each inequality corresponds to a **half-space**, and the feasible region is the **intersection of half-spaces**. In two dimensions, the boundaries are lines; in higher dimensions, they are hyperplanes.

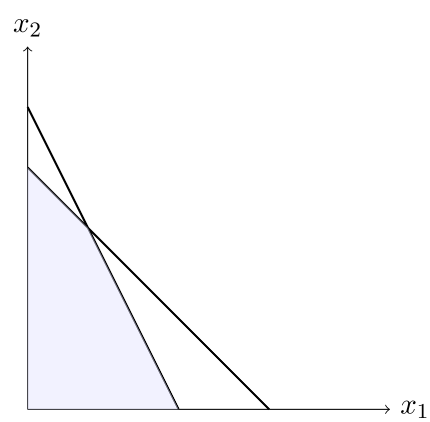
Key observations:

- If the region is bounded and nonempty, the LP has an optimal solution at one of the corner points (vertices).
- If the region is unbounded, it may or may not have a finite optimum.
- If the feasible region is empty, the LP is **infeasible**.

8 Graphic Method

Maximize $z = 3x_1 + 2x_2$
Subject to:

$$x_1 + x_2 \le 4, \quad 2x_1 + x_2 \le 5, \quad x_1, x_2 \ge 0$$



Vertices: $(0,0), (0,4), (1,3), (2.5,0)$
Best at: $(1,3)$, $z = 3(1) + 2(3) = 9$

9 Review and Practice

Problem 1: Solve the following systems of linear equations using the Gaussian elimination method:

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 54 \\ 2x_1 + 3x_2 + x_3 = 48 \\ 3x_1 + x_2 + 2x_3 = 60 \end{cases}$$

$$\begin{cases} 2x_1 + x_2 - x_3 = 1 \\ -3x_1 - x_2 + 2x_3 = -4 \\ -2x_1 + x_2 + 2x_3 = -3 \end{cases}$$

Problem 2: Are the following vectors linearly independent? Why?

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Problem 3: Compute the determinant of each of the following matrices:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix}$$

Problem 4: Solve the following system of linear equations by using the inverse matrix:

$$\begin{cases} 2x_1 + 3x_2 = 8 \\ x_1 + 2x_2 = 5 \end{cases}$$

Problem 5: Find the rank of each of the following matrices:

$$C = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{bmatrix}$$

Problem 6: Convert the following to standard form:

$$\begin{aligned} \text{Maximize} \quad & z = 3x_1 + 5x_2 \\ \text{Subject to} \quad & x_1 + 2x_2 \leq 8 \\ & 4x_1 + x_2 \geq 4 \\ & x_1 \text{ unrestricted}, \quad x_2 \geq 0 \end{aligned}$$

10 Conclusion

- Matrix theory is foundational to LP
- Determinants, rank, and linear systems tie directly to LP feasibility
- Next steps: simplex method, duality, sensitivity analysis

Reference

Shu-Cherng Fang and Sarat Puthenpura, "Linear Optimization and Extensions: Theory and Algorithms." Prentice Hall 1993.